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Power-Law Falloff in the Kosterlitz–Thouless Phase of a Two-Dimensional Lattice Coulomb Gas

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We give a rigorous proof of power-law falloff in the Kosterlitz–Thouless phase of a two-dimensional Coulomb gas in the sense that there exists a critical inverse temperature $\bar{\beta}$ and a constant $\Theta > 0$ such that for all $\beta > \bar{\beta}$ and all external charges $\xi \in \mathbf{R}$ we have $G_\xi(x) \leqslant C/|x|^{\Theta\beta\eta^2}$, where $G_\xi(x)$ is the two-point external charges correlation function, $\eta = \mathrm{dist}(\xi, \mathbf{Z})$, and $C = C(\beta, \bar{\eta}) < \infty$ for $\bar{\eta} = \mathrm{dist}(\xi, \mathbf{Z} - \{0\}) > 0$. In the case of a hard-core or standard Coulomb gas with activity z, we may choose $\bar{\beta} = \bar{\beta}(z)$ such that $\bar{\beta}(z) \to 24\pi$ as $z \searrow 0$.

KEY WORDS: Coulomb gas; Kosterlitz–Thouless phase.

1. INTRODUCTION

A two-dimensional (lattice) Coulomb gas is a system of classical particles with electric charges ± 1, whose possible positions range over a finite array of sites $\Lambda \subset \mathbf{Z}^2$, interacting via a two-body Coulomb potential.

A configuration of the gas is given by a function $q_\Lambda = \{q(x)\}_{x \in \Lambda}$ with values in $\mathbf{Z}$, $q(x)$ being the total charge concentrated at x. To each configuration we associate a Boltzmann factor $\bar{e}^{\beta E(q_\Lambda)}$, where β is the inverse temperature and $E(q_\Lambda)$ is the total electrostatic energy,

$$E(q_\Lambda) = \tfrac{1}{2}\langle q_\Lambda, (-\Delta)^{-1} q_\Lambda \rangle$$

where Δ is the finite-difference Laplacian on $\mathbf{Z}^2$ (we take "free" boundary conditions) and q_Λ was extended to $\mathbf{Z}^2$ by $q(x) = 0$ for $x \notin \Lambda$.

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The equilibrium state of the system is defined by the usual Gibbs measure $d\mu_\Lambda$ on the space of all configurations on Λ, given by

$$d\mu_\Lambda(q_\Lambda) = Z_\Lambda^{-1} e^{-\beta E(q_\Lambda)} \prod_{x \in \Lambda} d\lambda(q(x))$$

where

$$Z_\Lambda = \int e^{-\beta E(q_\Lambda)} \prod_{x \in \Lambda} d\lambda(q(x))$$

is the partition function. Here $d\lambda$, the "*a priori*" distribution, is a positive (not necessarily finite) measure on $\mathbf{Z}$.

Notice that the measure $d\mu_\Lambda$ is concentrated on the neutral configurations, i.e., $\sum_{x \in \Lambda} q(x) = 0$, since $E(q_\Lambda) = \infty$ if q_Λ is not neutral.

A thermodynamic limit as $\Lambda \nearrow \mathbf{Z}^2$ can always be constructed by a compactness argument.

The usual "*a priori*" distribution $d\lambda$ have densities λ given by:

1. The hard-core gas:

$$\lambda(q) = \begin{cases} 1 & q = 0 \\ z/2 & q = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

where $z > 0$ is called the activity.

2. The standard gas:

$$\lambda(q) = \frac{1}{2\pi} \int_0^{2\pi} e^{z \cos \Theta} \cos q\Theta \, d\Theta, \qquad q \in \mathbf{Z}$$

with activity $z > 0$.

3. The Villain gas:

$$\lambda(q) = 1 \qquad \text{for all} \quad q \in \mathbf{Z}$$

More generally, we will always require λ to satisfy:

(a) $\lambda(q) = \lambda(-q)$.

(b) $|\lambda(q)| \leqslant Ce^{\nu q^2}$ for some $\nu \geqslant 0$.

Such λ will be called *typical.*

We will be interested in the behavior of the external charges correlation function defined by

$$G_{\xi,\Lambda}(x) = \frac{Z_{\xi,\Lambda}(x)}{Z_\Lambda}$$

where

$$Z_{\xi,\Lambda}(x) = \int e^{-\beta E(q_\Lambda + \xi(\delta_0 - \delta_x))} \prod_{y \in \Lambda} d\lambda(q(y))$$

and $\xi \in \mathbf{R}$. By $G_\xi(x)$ we will denote a thermodynamic limit.

Using Jensen's inequality in the q variables, it is not hard to see that[1]

$$G_\xi(x) \geqslant \frac{C_{\beta,\xi}}{|x|^{\beta\xi^2/2\pi}} \tag{1.1}$$

for some $0 < C_{\beta,\xi} < \infty$.

When screening occurs (e.g., at high temperature) Brydges and Federbush[2] and Yong[9] have shown that

$$G_\xi(x) \to L > 0 \qquad \text{as} \quad x \to \infty$$

exponentially fast.

Fröhlich and Spencer[3] established the existence of a Kosterlitz–Thouless transition from a high-temperature phase to a low-temperature phase characterized by scaling and power-law falloff of correlations. They proved that, for β sufficiently large, Debye screening does not occur and

$$G_\xi(x) \leqslant \frac{C}{|x|^{\beta'}} \tag{1.2}$$

where $\beta' = \beta'(\beta, \xi)$, with $\lim_{\beta \to \infty} \beta' = \infty$.

Fröhlich and Spencer reduced the study of a general Coulomb gas to the study of the hard-core-type Coulomb gas. For such they proved that (see Section 5.1 in ref. 3)

$$G_\xi(x) \leqslant \frac{2}{|x|^{c\beta\eta^2 - b}} \tag{1.3}$$

for $\beta > b/c\eta^2$, where $\eta = \min(\xi, 1 - \xi)$ and $0 < \xi < 1$.

In this article we prove a power-law falloff for $G_\xi(x)$ with the same functional dependence on β and the fractional part of ξ as the lower bound (1.1). The critical β above which we get power-law falloff is independent of the external charges $\pm\xi$. We also show that for the hard-core or standard Coulomb gas this critical β is at most 24π in the low-activity limit. More precisely, we have:

Theorem 1.1. Suppose the *a priori* distribution λ is typical. Then there exist $\bar{\beta} = \bar{\beta}(\lambda) < \infty$ and $\Theta = \Theta(\lambda) > 0$ such that for all $\beta > \bar{\beta}$ we have

$$G_\xi(x) \leqslant \frac{C}{|x|^{\Theta\beta\eta^2}}$$

for some $C = C(\beta, \bar{\eta})$, where $\eta = \text{dist}(\xi, \mathbf{Z})$ and $\bar{\eta} = \text{dist}(\xi, \mathbf{Z} - \{0\})$, and $C_\gamma(\beta) = \sup_{\gamma < \bar{\eta} \leqslant 1} C(\beta, \bar{\eta}) < \infty$ for any $\gamma > 0$ and $\beta > \bar{\beta}$, with $\lim_{\beta \to \infty} C_\gamma(\beta) = 1$.

Theorem 1.2. For the hard-core or standard Coulomb gas with activity z, we may choose $\bar{\beta} = \bar{\beta}(z)$ in such a way that $\bar{\beta}(z) \to 24\pi$ as $z \searrow 0$.

Our proofs use ideas developed for the hierarchical model by Marchetti and Perez[4,5] combined with the main ingredients of the Fröhlich–Spencer proof. As in ref. 3, expectations in the two-dimensional Coulomb gas are written as convex combinations of expectations in diluted gases of neutral multipoles of variables sizes and falloff is extracted from charged multipoles by an analytic continuation argument. But our proof is organized in a different manner, so we avoid their combinatorial estimates involving many differents scales. In this we were influenced by the work of von Dreifus,[6] Spencer,[7] and von Dreifus and Klein.[8]

This paper is organized as follows. In Section 2 the partition function of a *typical* Coulomb gas is rewritten as a convex combination of *regular* partition functions (Theorem 2.3). This is the initial step which gives us the starting conditions for the inductive procedure presented in Section 3 (see Theorem 3.1). In Section 4 this inductive procedure is modified to treat the external charges partition function, from which we extract the power-law falloff (Theorem 4.3). We consider this the most important section in this work; here we use ideas from the hierarchical approximation.[5] In Section 5 we perform the cancellations necessary to show decay for the external charges correlation function.

2. THE FIRST STEP

Following ref. 3, we start by rewriting the Coulomb gas in the sine-Gordon representation. In this representation,

$$Z_\Lambda = \int \prod_{x \in \Lambda} \hat{\lambda}(\phi(x))\, d\mu_\beta(\phi) \tag{2.1}$$

where $d\mu_\beta$ is the Gaussian measure with covariance $\beta(-\Delta)^{-1}$, and

$$\hat{\lambda}(\phi) = \sum_{q \in \mathbf{Z}} e^{iq\phi} \lambda(q)$$

Similarly,

$$Z_{\xi, \Lambda}(x) = \int e^{i\xi\phi(\delta_0 - \delta_x)} \prod_{y \in \Lambda} \hat{\lambda}(\phi(y))\, d\mu_\beta(\phi) \tag{2.2}$$

Again following ref. 3, we choose $\{\zeta_q\}_{q=1}^{\infty}$ such that $\sum_{q=1}^{\infty} \zeta_q = 1$, ζ_q depending on $\lambda(q)$ in a way to be specified later. We write

$$\hat{\lambda}(\phi) = 1 + 2 \sum_{q=1}^{\infty} \lambda(q) \cos(q\phi)$$
$$= \sum_{q=1}^{\infty} \zeta_q (1 + 2\zeta_q^{-1} \lambda(q) \cos(q\phi))$$

Thus,

$$\prod_{x \in \Lambda} \hat{\lambda}(\phi(x)) = \sum_{q_\Lambda} \zeta_{q_\Lambda} \prod_{x \in \Lambda} \{1 + 2\zeta_{q(x)}^{-1} \lambda(q(x)) \cos[q(x)\,\phi(x))]\}$$

where $\zeta_{q_\Lambda} = \prod_{x \in \Lambda} \zeta_{q(x)}$, the sum being over all configurations in Λ with values in $\mathbf{N}$.

We now need to introduce some notation. For $x = (x_1, x_2) \in \mathbf{Z}^2$, we will write $|x|$ for $|x|_\infty = \max(|x_1|, |x_2|)$. By $B(x, L)$ we will denote the square in $\mathbf{Z}^2$ centered at x with side L, i.e.,

$$B(x, L) = \{ y \in \mathbf{Z}^2; |y - x| < L/2 \}$$

Now let Λ be a large square centered at the origin; we define

$$\hat{\Lambda} = \Lambda \cap 3\mathbf{Z}^2$$

and for $y \in \hat{\Lambda}$, we let

$$B_0(y) = B(y, 3)$$

Clearly $\Lambda = \bigcup_{y \in \hat{\Lambda}} B_0(y)$. Hence, for a fixed q_Λ,

$$\prod_{x \in \Lambda} \{1 + z(q(x)) \cos[q(x)\,\phi(x)]\}$$
$$= \prod_{y \in \hat{\Lambda}} \prod_{u \in B_0(y)} \{1 + z(q(u)) \cos[q(u)\,\phi(u)]\}$$

where $z(q) = 2\zeta_q^{-1} \lambda(q)$.

For fixed $y \in \hat{\Lambda}$ we will write the above product as a convex combination of similar factors corresponding to multipoles of variable size. For that we will need the following lemma, which extends a result of ref. 3.

Lemma 2.1. Let I be an index set with N elements, and let $z_i \geq 0$, $\phi_i \in \mathbf{R}$ be given for each $i \in I$. Then,

$$\prod_{i \in I} (1 + z_i \cos \phi_i) = \sum_{\sigma \in \mathscr{G}(I)} c_\sigma [1 + z_\sigma \cos \phi(\sigma)] \tag{2.3}$$

where $\mathscr{G}(I)=\{\sigma: I\to\{0, 1, -1\};\ \sigma$ not identically zero$\}$,

$$\phi(\sigma)=\sum_{i\in I}\sigma_i\phi_i$$

and

$$z_\sigma=\prod_{i\in I}(b_I z_i)^{|\sigma_i|} \tag{2.4}$$

where b_I is a constant depending only on N such that

$$0<b_I\leqslant\frac{N}{\log 2} \tag{2.5}$$

and

$$0<c_\sigma, \qquad \sum_{\sigma\in\mathscr{G}(I)}c_\sigma=1$$

Lemma 2.1 is proved in Appendix A.

So, let $y\in\hat{\Lambda}$; by the lemma,

$$\prod_{u\in B_0(y)}\{1+z(q(u))\cos[q(u)\,\phi(u)]\}=\sum_{\sigma_y\in\mathscr{G}(B_0(y))}c_{\sigma_y}[1+z_{\sigma_y}\cos\phi(\rho_{\sigma_y})]$$

where

$$\rho_{\sigma_y}(u)=\sigma_y(u)\,q(u)$$

$$\phi(\rho)=\sum_{u\in B_0(y)}\phi(u)\,\rho(u)$$

and

$$0<z_{\sigma_y}\leqslant\prod_{u\in B_0(y)}\left[\frac{9}{\log 2}z(u)\right]^{|\sigma_y(u)|} \tag{2.6}$$

Thus, Z_Λ can be written as a convex combination of partition functions of the type

$$Z_{\hat{\Lambda}}=\int\prod_{y\in\hat{\Lambda}}[1+z_y\cos\phi(\rho_y)]\,d\mu_\beta(\phi) \tag{2.7}$$

where $\rho_y: B_0(y)\to\mathbf{Z}$, and

$$0<z_y\leqslant\prod_{\substack{u\in B_0(y):\\ \rho_y(u)\neq 0}}\left[\frac{9}{\log 2}\,\frac{2\lambda(\rho_y(u))}{\zeta_{\rho_y(u)}}\right] \tag{2.8}$$

We now want to improve the estimate on the activity z_y by extracting a self-energy term as in ref. 3, Lemma 4.2. Here we need a slight extension.

Lemma 2.2. Let $\rho: \mathbf{Z}^2 \to \mathbf{R}$ with compact support and let $G(\phi)$ be a functional independent of $\{\phi(x); x \in \operatorname{supp} \rho\}$. Then

$$\int [\exp i\phi(\rho)]\, G(\phi)\, d\mu_\beta(\phi)$$

$$= \exp\left[-\frac{\beta}{16}\sum_x \rho(x)^2\right] \int [\exp i\phi(\bar{\rho})]\, G(\rho)\, d\mu_\beta(\phi) \tag{2.9}$$

where $\bar{\rho} = \rho + \frac{1}{8}\Delta\rho$.

The proof is the same as that of Lemma 5.3 in ref. 10 using the imaginary shift $\phi \to \phi + i\frac{1}{8}\beta\rho$. Just use $\|\Delta\| = 8$.

We would like to apply Lemma 2.2 to each ρ_y in (2.7). We cannot do it directly because even if $\operatorname{supp} \rho_y \subset B_0(y)$, $\operatorname{supp} \bar{\rho}_y \subset B(y, 5/2)$, but not necessarily in $B_0(y)$. If we disregarded this technical problem, applying Lemma 2 to each ρ_y in (2.7) would give

$$Z_{\hat{\Lambda}} = \int \prod_{y \in \hat{\Lambda}} [1 + \bar{z}_y \cos \phi(\bar{\rho}_y)]\, d\mu_\beta(\phi)$$

with $\bar{z}_y = z_y \exp[-\frac{1}{16}\beta \sum_x \rho_y(x)^2]$.

But since $\bar{\rho}_y$ and $\rho_{y'}$ could overlap, with $y \neq y'$, we must do some contorsions. In each $B_0(y)$ we pick u_y such that

$$\rho_y(u_y)^2 \geqslant \frac{1}{9} \sum_{u \in B_0(y)} \rho_y(u)^2$$

[recall $|B_0(y)| = 9$].

Let us define the equivalence

$$y \sim y' \Leftrightarrow |u_y - u_{y'}| = 1$$

and let $Y_1, \ldots, Y_J$ denote the distinct equivalence classes. Notice we always have $|Y_i| = 1, 2, 3$, or 4. For each Y_i we use Lemma 2.1 for the ρ_y with $y \in Y_i$, and we define

$$B_0(Y_i) = \bigcup_{y \in Y_i} B_0(y)$$

The end result is that $Z_{\hat{\Lambda}}$ can be expanded as a convex combination of expressions of the form

$$\int \prod_Y [1 + z_Y \cos \phi(\rho_Y)]\, d\mu_\beta(\phi) \tag{2.10}$$

where supp $\rho_Y \subset B_0(Y)$ and

$$z_Y \leqslant \left(\frac{4}{\log 2}\right)^4 \prod_{\substack{u \in B_0(Y):\\ \rho_Y(u) \neq 0}} \left[\frac{9}{\log 2} \frac{2\lambda(\rho_Y(u))}{\zeta_{\rho_Y(u)}}\right] \tag{2.11}$$

Now, if $U_Y = \{u_y, y \in Y\}$, we have $d(U_Y, U_{Y'}) > 1$ if $Y \neq Y'$. We can now apply Lemma 2.2 to each ρ_Y restricted to U_Y in (2.10), obtaining

$$\int \prod_Y \{1 + \bar{z}_Y \cos[\phi(\bar{\rho}_Y)]\, d\mu_\beta(\phi)\} \tag{2.12}$$

where

$$\bar{z}_Y = \left\{\exp\left[-\frac{1}{144}\beta \sum_u \rho_Y(u)^2\right]\right\} z_Y \tag{2.13}$$

and

$$\bar{\rho}_Y = \rho_Y + \tfrac{1}{8}\Delta(\rho_Y \mid U_Y)$$

Now, set

$$K_0(\beta) = \left(\frac{4}{\log 2}\right)^4 \frac{18}{\log 2} \sup_{q = 1, 2, \ldots} \left[\frac{\lambda(q)}{\zeta_q} e^{-\beta q^2/144}\right] \tag{2.14}$$

It follows from (2.11) and (2.14) that

$$\bar{z}_Y \leqslant K_0(\beta) \qquad \text{for all } Y \tag{2.15}$$

Moreover, since λ is typical, we have $\lim_{\beta \to \infty} K_0(\beta) = 0$.

We will always pick β such that $K_0(\beta) < 1$.

We now need some definitions. A charge density is a function $\rho: \mathbf{Z}^2 \to \mathbf{R}$; ρ is said to be localized on a square $B(y, L)$, or to be a charge in $B(y, L)$, if

$$\text{supp } \rho \subset B\left(y, \frac{4}{3} L\right) \equiv \bar{B}(y, L)$$

A weighted charge density is a pair (ρ, z), where ρ is a charge density with an activity $z \geqslant 0$. From now on all our charges will be weighted; we will write ρ also for the pair (ρ, z), and will use $z(\rho)$ for the corresponding activity. Notice that (2.12) is written in terms of weighted charge densities whose activities satisfy the bound (2.15).

In (2.12) our charges are localized on sets $B_0(Y)$ that may have four different shapes. We will rectify this by going to a larger scale L_1.

So let $L_1 = 3^{n_1}$, where n_1 is an integer $\geqslant 3$, let $\Lambda^{(1)} = \Lambda \cap L_1\mathbf{Z}^2$, and let $B_1(y) = B(y, L_1)$. We pick a function $F\colon \{Y\} \to \Lambda^{(1)}$ such that

$$F(Y) = y \Rightarrow B_0(Y) \subset \bar{B}_1(y)$$

Now, for each $y \in \Lambda^{(1)}$ we use Lemma 2.1 to get

$$\prod_{Y\colon F(Y)=y} [1 + \bar{z}_Y \cos\phi(\bar{\rho}_Y)] = \sum_{\sigma \in \mathscr{G}(F^{-1}(y))} c_\sigma [1 + z(\rho_\sigma) \cos\phi(\rho_\sigma)]$$

where $0 < c_\sigma$, $\sum_\sigma c_\sigma = 1$, and ρ_σ is localized on $B_1(y)$ with

$$z(\rho_\sigma) \leqslant \left(\frac{4}{9} L_1\right)^2 \frac{K_0(\beta)}{\log 2} \equiv K_1(L_1, \beta) \tag{2.16}$$

if $K_1(L_1, \beta) < 1$.

Thus, (2.12) can be rewritten as a convex combination of expressions of the form

$$\int \prod_{y \in \Lambda^{(1)}} [1 + z(\rho_y) \cos\phi(\rho_y)] \, d\mu_\beta(\phi) \tag{2.17}$$

where ρ_y is localized on $B_1(y)$ and $z(\rho_y) \leqslant K_1(L_1, \beta)$.

We have thus proven the following theorem.

Theorem 2.3. Let $L_1 = 3^{n_1}$ with $n_1 \geqslant 3$. Then, if $K_1(L_1, \beta) < 1$, the partition function of a two-dimensional Coulomb gas can always be written as a convex combination of partition functions of the form (2.17) with activities satisfying (2.16).

Equations (2.16) and (2.17) will give us the initial conditions for the inductive procedure we describe in the next section.

The external charges partition function (2.2) can be treated in a similar way with some modifications. We postpone it to Section 4.

3. THE INDUCTIVE PROCEDURE FOR THE PARTITION FUNCTION

Let us fix $\alpha > 1$, $n_1 \geqslant 3$; we set $L_1 = 3^{n_1}$ and, for $k = 1, 2, 3,...,$ $L_{k+1} = 3^{n_{k+1}}$, where $n_{k+1} = [\alpha n_k]$.

We set $\Lambda^{(k)} = \Lambda \cap L_k\mathbf{Z}^2$ and define $B_k(y) \equiv B(y, L_k)$ for $y \in \Lambda^{(k)}$; $\bar{B}_k(y) = B(y, \frac{4}{3}L_k)$. We will also need $B_k^{(k')}(y) = B_k(y) \cap L_{k'}\mathbf{Z}^2$ for $k' \leqslant k$.

We will always take Λ to be a square centered at 0, say $\Lambda = B(0, R)$, and we pick N such that $L_{N-1} < R \leqslant L_N$. Notice that $\Lambda^{(N)} = \{0\}$.

Let us fix a scale k, a number $t>0$, and $y \in \Lambda^{(k)}$. A weighted charge density $\rho=(\rho, z(\rho))$ is (k, y, t)-admissible if:

(i) ρ is localized on $B_k(y)$.

(ii) $z(\rho) \leqslant L_k^{-t}$.

A collection $\mathcal{N}_{(k,y,r)}$ of neutral weighted charge densities will be called a (k, y, r)-sparse neutral ensemble if:

(i) For $k=1$, $\mathcal{N}_{(1,y,r)}=\phi$.

(ii) For $k=2, 3,\ldots$, we have

$$\mathcal{N}_{(k,y,r)}=\left[\bigcup_{y' \in B_k^{(k-1)}(y)} \mathcal{N}_{(k-1,y',r)}\right] \cup \{(\rho, z)\}$$

where (ρ, z) is localized on $B_{k-1}(y')$ for some $y' \in B_k^{(k-1)}(y)$, with $z \leqslant [8/(\log 2)^4]\, L_{k-1}^{-r}$ and each $\mathcal{N}_{(k-1,y',r)}$ is a $(k-1, y', r)$-sparse neutral ensemble.

Given $\mathcal{N}_{(k,y,r)}$, let

$$F(\mathcal{N}_{(k,y,r)}, \phi)=\prod_{\rho \in \mathcal{N}_{(k,y,r)}} [1+z(\rho) \cos \phi(\rho)]$$

Given a scale k, a (k,r)-regular charge assignment is a collection $\{\mathcal{N}_{(k,y,r)}, (\rho_y, z_y)\}_{y \in \Lambda^{(k)}}$ where each $\mathcal{N}_{(k,y,r)}$ is a (k, y, r)-sparse neutral ensemble and each (ρ_y, z_y) is a $[k, y, r+2(\alpha-1)]$-admissible charge density.

A (k, r)-regular partition function is a partition function of the form

$$Z_{(k,r)}=\int \prod_{y \in \Lambda^{(k)}} F(\mathcal{N}_{(k,y,r)}; \phi)[1+z(\rho_y) \cos \phi(\rho_y)]\, d\mu_\beta(\phi) \qquad (3.1)$$

where $\{\mathcal{N}_{(k,y,r)}, (\rho_y, z_y)\}$ is a (k, r)-regular charge assignment.

In this language, Theorem 2.3 just states that for a choice of parameters such that $K_1(L_1, \beta) \leqslant L_1^{-(r+2(\alpha-1))}$, the partition function Z_Λ given by (2.1) is a convex combination of $(1, r)$-regular partition functions.

In other words, Theorem 2.3 gives the initial step in the inductive procedure of the following theorem.

Theorem 3.1. Let $3/2<\alpha<2$, $2\alpha(\alpha-1)/(2-\alpha)<r<\beta\delta/2-2\alpha$, with $\delta^{-1}=8+O(L_1^{3-2\alpha})$. Suppose $K_1(L_1, \beta)<L_1^{-(r+2(\alpha-1))}$. Then, if L_1 is large enough, the Coulomb gas partition function Z_Λ can always be written as a convex combination of (k, r)-regular partition functions for any $k=1, 2,\ldots, N$.

In view of Theorem 2.3, Theorem 3.1 follows from the following result.

Lemma 3.2. Let α, r, L_1 be as above. Let $k = 1, 2,..., N-1$. Then, if L_1 is large enough, any (k, r)-regular partition function can be written as a convex combination of $(k+1, r)$-regular partition functions.

Proof. Let $k \in \{1, 2,..., N-1\}$ and let $\{\mathcal{N}_{(k,y,r)}, (\rho_y, z_y)\}_{y \in \Lambda^{(k)}}$ be a (k, r)-regular charge assignment. Let $Z_{(k,r)}$ be given by (3.1).

Given $u \in \Lambda^{(k+1)}$, we define

$$\tilde{\mathcal{N}}_{(k+1,u,r)} = \bigcup_{y \in B_{k+1}^{(k)}(u)} \mathcal{N}_{(k,y,r)}$$

Clearly, $\tilde{\mathcal{N}}_{(k+1,u,r)}$ is a $(k+1, u, r)$-sparse neutral ensemble and

$$Z_{(k,r)} = \int \prod_{u \in \Lambda^{(k+1)}} F(\tilde{\mathcal{N}}_{(k+1,u,r)}; \phi) \prod_{y \in \Lambda^{(k)}} [1 + z_y \cos \phi(\rho_y)] \, d\mu_\beta(\phi)$$

Using Lemma 2.1, this can be written as a convex combination of partition functions of the form

$$\int \prod_{u \in \Lambda^{(k+1)}} F(\tilde{\mathcal{N}}_{(k+1,u,r)}; \phi) \prod_{u \in \Lambda^{(k+1)}} [1 + z_u^* \cos \phi(\rho_u^*)] \, d\mu_\beta(\phi) \quad (3.2)$$

where each (ρ_u^*, z_u^*) is of the form

$$\rho_u^* = \sum_{y \in \Lambda_{k+1}^{(k)}(u)} \sigma_y \rho_y$$

for some $\sigma \in \mathcal{G}(\Lambda_{k+1}^{(k)}(u))$ and

$$z_u^* \leqslant \left[\frac{1}{\log 2}\left(\frac{L_{k+1}}{L_k}\right)^2 \frac{1}{L_k^{r+2(\alpha-1)}}\right]^{\Sigma_y |\sigma_y|} \leqslant \left[\frac{1}{\log 2}\frac{1}{L_k^r}\right]^{\Sigma_y |\sigma_y|} \quad (3.3)$$

To propagate our bound on the activities to the next scale, we will need, in some cases, to extract a self-energy term as in refs. 3 and 10. This will be done by the following lemma (see Lemma 5.4 in ref. 10).

Lemma 3.3. Let $\{\tilde{\mathcal{N}}_{(k+1,u,r)}, (\rho_u^*, z_u^*)\}_{u \in \Lambda^{(k+1)}}$ be as above. Suppose that for some $u_0 \in \Lambda^{(k+1)}$, $\rho_{u_0}^* = \rho_{y_0}$ for some $y_0 \in B_{k+1}^{(k)}(u_0)$ and that

$$B(y_0, \tfrac{1}{3}L_{k+1}) \cap \operatorname{supp} \rho_u^* = \phi \quad (3.4)$$

for all $u \in \Lambda^{(k+1)}$, $u \neq u_0$. Then, if $\alpha > 3/2$, we have, for given $\kappa > 0$,

$$\int e^{\pm i\phi(\rho^*_{u_0})} \prod_{u \in \Lambda^{(k+1)}} F(\tilde{\mathcal{N}}_{(k+1,u,r)}; \phi) \prod_{\substack{u \in \Lambda^{(k+1)} \\ u \neq u_0}} [1 + z^*_u \cos \phi(\rho^*_u)]\, d\mu_\beta(\phi)$$

$$= Y^{(k)} \int e^{\pm i\phi(\bar{\rho}^*_{u_0})} \prod_{u \in \Lambda^{(k+1)}} F(\tilde{\mathcal{N}}_{(k+1,u,r)}; \phi)$$

$$\times \prod_{\substack{u \in \Lambda^{(k+1)} \\ u \neq u_0}} [1 + z^*_u \cos \phi(\rho^*_u)]\, d\mu_\beta(\phi) \tag{3.5}$$

where

$$\left(\frac{L_{k+1}}{3L_k}\right)^{-\beta\kappa q^2} \leqslant Y^{(k)} \leqslant \left(\frac{L_{k+1}}{3L_k}\right)^{-\beta\kappa q^2(1-\kappa/2\delta)}$$

with $q \equiv Q(\rho^*_u) = \sum_{y \in \bar{B}_k(u)} \rho^*_u(y)$, $\delta^{-1} = [\delta(\alpha, L_1)]^{-1} = 8 + O(L_1^{3-2\alpha})$ is independent of k and $\bar{\rho}^*_{u_0}$ is a charge density localized on $B(y_0, \frac{1}{3}L_{k+1})$ such that $Q(\bar{\rho}^*_{u_0}) = Q(\rho^*_{u_0})$.[3]

The lemma will be proved in Appendix B.

Lemma 3.3 requires (3.4), which may not be true in the situation where we would like to use the lemma. To avoid this problem, we use Lemma 2.1 to establish (3.4) for all ρ^*_u of the form $\rho^*_u = \rho_y$ for some $y \in B^{(k)}_{k+1}(u)$ if it was not already satisfied.

Clearly, that gives (3.2) as a convex combination of expressions of the same type, but with (3.4) holding, with

$$\rho^*_u = \sum_y \tau_y \rho_y \qquad \text{for some} \quad \tau \in \mathcal{G}(\bar{B}_{k+1}(u))$$

and

$$z^*_u \leqslant \left[\left(\frac{2}{\log 2}\right)^3 \left(\frac{1}{\log 2} \frac{1}{L^r_k}\right)\right]^{\sum_y |\tau_y|} \tag{3.6}$$

We did not change the $\tilde{\mathcal{N}}_{(k+1,u,r)}$.

We now consider several cases.

(i) $\sum_y |\tau_y| \geqslant 2$. In this case we define $\mathcal{N}_{(k+1,u,r)} = \tilde{\mathcal{N}}_{(k+1,u,r)}$ and notice that (3.6) gives

$$z^*_u \leqslant \frac{1}{L^{r+2(\alpha-1)}_{k+1}}$$

if $r > 2\alpha(\alpha-1)/(2-\alpha)$ and L_1 is sufficiently large.

[3] The proof of Lemma 3.3 can be slightly modified to obtain $\delta^{-1} = 2\pi + O(L_1^{3-2\alpha})$; see Appendix B.

We define $(\rho_u, z_u) = (\rho_u^*, z_u^*)$.

(ii) $\sum_y |\tau_y| = 1$. Here we must consider two subcases:

(iia) $Q(\rho_u^*) = 0$. We define $\mathcal{N}_{(k+1,u,r)} = \tilde{\mathcal{N}}_{(k+1,u,r)} \cup \{(\rho_u^*, z_u^*)\}$ and notice that it is a $(k+1, u, r)$-sparse neutral ensemble by (3.6).

(iib) $Q(\rho_u^*) \neq 0$. We define $\mathcal{N}_{(k+1,u,r)} = \tilde{\mathcal{N}}_{(k+1,u,r)}$. Now recall $\cos\Theta = \frac{1}{2}(e^{i\Theta} + e^{-i\Theta})$, and use Lemma 3.3 (here we choose $\kappa = \delta$) to replace (ρ_u^*, z_u^*) by (ρ_u, z_u), where $\rho_u = \bar{\rho}_u^*$ and

$$z_u \leqslant \left(\frac{L_{k+1}}{3L_k}\right)^{-\beta(\delta/2)} z_u^* \leqslant \frac{1}{L_{k+1}^{r+2(\alpha-1)}}$$

if $r < \beta\delta/2 - 2\alpha$ for L_1 sufficiently large.

This finishes the proof of Lemma 3.2 and hence also of Theorem 3.1.

4. THE EXTERNAL CHARGES PARTITION FUNCTION

We now show how to modify the procedure of Sections 2 and 3 to treat $Z_{\xi,\Lambda}(x)$ given by (2.2).

So, given $x \in \mathbf{Z}^2$, let us choose N_0 such that $L_{N_0} < |x| < L_{N_0+1}$. Without loss of generality, $N_0 > 1$. We want to prove the analogue of Theorem 3.1 for $Z_{\xi,\Lambda}(x)$. We start by showing the analogue of Theorem 2.3. Note that in the same way we obtain the analogue of (2.7),

$$Z_{\xi,\hat{\Lambda}}(x) = \int e^{i\xi\phi(\delta_0 - \delta_x)} \prod_{y \in \hat{\Lambda}} [1 + z_y \cos\phi(\rho_y)]\, d\mu_\beta(\varphi) \tag{4.1}$$

with the same (ρ_y, z_y).

We must be careful, though, when we apply Lemma 2.2. Since $|x| > L_1/2$, 0 and x do not belong to the same $B_0(Y)$. We have, of course, $0 \in B_0(Y_0)$ and $x \in B_0(Y_x)$ for some $Y_0, Y_x \in \hat{\Lambda}$. When we apply Lemma 2.2 we must treat ρ_{Y_0} and ρ_{Y_x} differently from the others. If $0 \in Y_0$ and $u_0 \neq 0$, we do as before. If, however, $u_0 = 0$, we must apply Lemma 2.2 to both

$$\exp[i\phi(\rho_{Y_0} + \xi\delta_0)] \qquad \text{and} \qquad \exp[-i\phi(\rho_{Y_0} - \xi\delta_0)]$$

The result is that $[1 + z_{Y_0} \cos\phi(\rho_{Y_0})]$ is replaced by

$$\left\{1 + \frac{z_{Y_0}^+}{2}\exp[i\phi(\rho_{Y_0}^+)] + \frac{z_{Y_0}^-}{2}\exp[-i\phi(\rho_{Y_0}^-)]\right\}$$

where [see (2.13)]

$$\begin{aligned} \rho_{Y_0}^+ &= \overline{\rho_{Y_0} + \xi\delta_0} - \xi\delta_0 \\ \rho_{Y_0}^- &= \overline{\rho_{Y_0} - \xi\delta_0} + \xi\delta_0 \end{aligned} \tag{4.2}$$

and

$$z^{\pm}_{Y_0} \leqslant \exp\left[-\frac{1}{144}\beta\bar{\eta}^2 \sum_x \rho_{Y_0}(x)^2\right]$$

where the extra factor [compare with (2.14)] comes from

$$(q \pm \xi)^2 \geqslant \bar{\eta}^2$$

for $q \geqslant 1$, $\bar{\eta} = \text{dist}(\xi, \mathbf{Z} - \{0\})$.

Thus, in any case, (2.12) is replaced by

$$\int \{\exp[i\xi\phi(\delta_0 - \delta_x)]\} \prod_{y=0,x} \left\{1 + \frac{z^+_{Y_y}}{2}\exp[i\phi(\rho^+_{Y_y})] + \frac{z^-_{Y_y}}{2}\exp[-i\phi(\rho^-_{Y_y})]\right\}$$
$$\times \prod_{Y \neq Y_0,\, Y_x} [1 + \bar{z}_Y \cos\phi(\bar{\rho}_Y)]\, d\mu_\beta(\phi) \qquad (4.3)$$

$\bar{z}_Y$, $\bar{\rho}$ are as in (2.13) for $Y \neq Y_0$, Y_x, $\rho^{\pm}_{Y_y}$, $y = 0, x$, being either as before or given by (4.2). Notice that $Q(\rho^+) = Q(\rho^-)$. If we now define [recall (2.14)]

$$K'_0(\beta, \xi) = K_0(\beta\bar{\eta}^2)$$

we always have that, for sufficient large β,

$$z^{\pm}_{Y_0}, z^{\pm}_{Y_x} \leqslant K'_0(\beta, \xi) \qquad (4.4)$$

We now go to the bigger scale L_1. We pick a function H: $\{Y\} \to \Lambda^{(1)}$ such that:

(i) $H(Y) = y \Rightarrow B_0(Y) \subset \overline{B_1}(y)$.

(ii) $H(Y_0) = 0$ and $H(Y_x) = x_1$, where $x_1 \in \Lambda^{(1)}$ and $x \in B_1(x_1)$.

Now, for each $y \in \Lambda^{(1)} - \{0, x_1\}$, we proceed as in Section 2. For $u_1 = 0, x_1$ we first apply Lemma 2.1 to

$$\prod_{\substack{H:\, H(Y) = \mu_1 \\ Y \neq Y_{\mu_1}}} [1 + \bar{z}_Y \cos\phi(\bar{\rho}_Y)] = \sum_{\sigma \in \mathcal{G}(H^{-1}(u_1) - \{Y_{u_1}\})} c_\sigma[1 + z(\rho_\sigma)\cos\phi(\rho_\sigma)]$$

with $z(\rho_\sigma)$ satisfying (2.16). We then apply a slightly different version of Lemma 2.1 to each term of the type shown in the following lemma.

Lemma 4.1. Let (ρ, z), $(\rho^{\pm}, z^{\pm})$ be weighted densities localized on $B(u)$, with z, $z^{\pm} \geqslant 0$ and $Q(\rho^+) = Q(\rho^-)$. Then

$$[1 + z\cos\phi(\rho)]\left(1 + \frac{z^+}{2}e^{i\phi(\rho^+)} + \frac{z^-}{2}e^{-i\phi(\rho^-)}\right)$$
$$= \sum_{\tau \in \mathcal{F}} c_\tau \left(1 + \frac{z^+_\tau}{2}e^{i\phi(\rho^+_\tau)} + \frac{z^-_\tau}{2}e^{-i\phi(\rho^-_\tau)}\right)$$

where

$$\mathscr{F}=\{\tau=(\tau_1,\tau_2): \tau_i\in\{0,1,-1\}, i=1,2 \text{ and } \tau\neq(0,0)\},\quad \rho_\tau^\pm=\tau_1\rho+\tau_2\rho^\pm$$

with $Q(\rho_\tau^+)=Q(\rho_\tau^-)$ and

$$z_\tau^\pm=(b_2z)^{|\tau_1|}(b_2z^\pm)^{|\tau_2|}$$

where $0<b_2<2/\log 2$ and $0<c_\tau$, $\sum_{\tau\in\mathscr{F}}c_\tau=1$.

The result is that (4.3) is given as a convex combination of expressions of the form

$$\int e^{i\xi\phi(\delta_0-\delta_x)}\prod_{u=0,x_1}\left(1+\frac{z_u^+}{2}e^{i\phi(\rho_u^+)}+\frac{z_\mu^-}{2}e^{-i\phi(\rho_u^-)}\right)$$

$$\times\prod_{y\in\Lambda^{(1)}-\{0,x_1\}}[1+z_y\cos\phi(\rho_y)]\,d\mu_\beta(\phi) \tag{4.5}$$

where ρ_y is localized on $B_1(y)$ for $y\in\Lambda^{(1)}-\{0,x_1\}$ with

$$z_y\leqslant K_1(L_1,\beta) \tag{4.5a}$$

[see (2.16) for definition] and $\rho_u^\pm$ is localized on $B_1(u)$, $u=0, x_1$ with

$$z_u^\pm\leqslant\frac{2}{(\log 2)^2}\left(\frac{4}{9}L_1\right)^2K'_0(\beta,\xi)\equiv K'_1(L_1,\beta,\xi) \tag{4.5b}$$

Notice that if $K'_1<1$, then $K_1<1$.

We have proved the following theorem analogous to Theorem 2.3.

Theorem 4.2. Let $L_1=3^{n_1}$ with $n_1\geqslant 3$. Then, if $K'_1<1$, the two-point external charges partition function $Z_{\xi,\Lambda}(x)$ can always be written as a convex combination of partition functions of the form (4.5) with activities satisfying (4.5a) and (4.5b).

Theorem 4.2 gives us the initial condition for the inductive procedure we describe below.

Given $y\in\Lambda$, let $y_k\in\Lambda^{(k)}$ be such that $y\in B_k(y_k)$.

We use this notation to distinguish those squares which contain the external charges. So, for $k\leqslant N_0$, $y=0$ or x, we have $y_k=0$ or x_k with $x\in B_k(x_k)$, and for $k>N_0$, $y_k=0$.

A collection $\mathscr{E}_{(k,y_k,r,s)}$ of neutral weighted charge densities will be called a (k, y_k, r, s)-sparse modified neutral ensemble if:

(i) For $k=1$, $\mathscr{E}_{(1,y_1,r,s)}=\phi$.

(ii) For $k = 2, \ldots, N_0$,

$$\mathscr{E}_{(k, y_k, r, s)} = \bigcup_{\substack{y \in B_k^{(k-1)}(y_k) \\ u \neq y_{k-1}}} \mathscr{N}_{(k-1, u, r)} \cup \mathscr{E}_{(k-1, y_{k-1}, r, s)} \cup \{(\rho^{\pm}, z^{\pm})\}$$

(iii) For $k = N_0 + 1$

$$\mathscr{E}_{(N_0+1, 0, r, s)} = \bigcup_{\substack{u \in B_{N_0+1}^{(N_0)}(0) \\ u \neq 0, x_{N_0}}} \mathscr{N}_{(N_0, u, r)} \bigcup_{y_{N_0} = 0, x_{N_0}} \mathscr{E}_{(N_0, y_{N_0}, r, s)} \cup \{(\rho^{\pm}, z^{\pm})\}$$

(iv) For $k > N_0 + 1$, we have as in (ii).

Here $(\rho^{\pm}, z^{\pm})$ is a weighted charge density localized on $B_{k-1}(y_{k-1})$ with $Q(\rho^+) = Q(\rho^-)$ and $z^{\pm} \leqslant 2^3 \cdot 3/(\log 2)^4 \cdot L_{k-1}^{-s}$. The $\mathscr{E}_{(k-1, y_{k-1}, r, s)}$ is a $(k-1, y_{k-1}, r, s)$-sparse modified neutral ensemble and each $\mathscr{N}_{(k-1, u, r)}$ is a $(k-1, u, r)$-sparse neutral ensemble as introduced in Section 3.

Given $\mathscr{E}_{(k, y_k, r, s)}$, we define the functional G as follows:

(i) $G(\mathscr{E}_{(1, y_1, r, s)}; \phi) = 1$ for $k = 1$.

(ii) We have

$$G(\mathscr{E}_{(k, y_k, r, s)}; \phi) = \prod_{\substack{u \in B_k^{(k-1)}(y_k) \\ u \neq y_{k-1}}} F(\mathscr{N}_{(k-1, u, r)}; \phi) \times G(\mathscr{E}_{(k-1, y_{k-1}, r, s)}; y)\{1 + \tfrac{1}{2}[z^+ e^{i\phi(\rho^+)} + z^- e^{-i\phi(\rho^-)}]\}$$

for $k \leqslant N_0$ and for $k > N_0 + 1$.

(iii) For $k = N_0 + 1$

$$G(\mathscr{E}_{(N_0+1, 0, r, s)}; \phi) = \prod_{\substack{u \in B_{N_0+1}^{(N_0)}(0) \\ u \neq 0, x_{N_0}}} F(\mathscr{N}_{(N_0, u, r)}; \phi) \times \left[\prod_{y_{N_0} = 0, x_{N_0}} G(\mathscr{E}_{(N_0, y_{N_0}, r, s)}; \phi) \right] \{1 + \tfrac{1}{2}[z^+ e^{i\phi(\rho^+)} + z^- e^{-i\phi(\rho^-)}]\}$$

Given a scale k, a (k, r, s)-modified charge assignment is a collection

$$\{\mathscr{N}_{(k, y, r)}, (\rho_y, z_y)\}_{y \in \Lambda^{(k)} - \{0, x_k\}} \cup \{\mathscr{E}_{(k, y, r, s)}, (\rho_y^{\pm}, z_y^{\pm})\}_{y = 0, x_k}$$

where for each $y \in \Lambda^{(k)} - \{0, x_k\}$, $\mathscr{N}_{(k, y, r)}$ is a (k, y, r)-sparse neutral ensemble and (z_y, ρ_y) is a $[k, y, r + 2(\alpha - 1)]$-admissible charge density. For $y = 0, x_k$, $\mathscr{E}_{(k, y, r, s)}$ is a (k, y, r, s)-sparse modified neutral ensemble and $(\rho_y^{\pm}, z_y^{\pm})$ is a (k, y, s)-admissible charge.

A (k, r, s)-regular two-point external charges partition function is a partition function of the form

$$Z_{\xi,\Lambda}^{(k,r,s)}(x)=\int \{\exp[i\xi\phi(w_k)]\} \prod_{y\in\Lambda^{(k)}-\{0,x_k\}} F(\mathcal{N}_{(k,y,r)};\phi)$$
$$\times [1+z(\rho_y)\cos\phi(\rho_y)]$$
$$\times \prod_{y_k=0,x_k} [W^{(k)}G(\mathscr{E}_{(k,y_k,r,s)};\phi)$$
$$\times (1+\tfrac{1}{2}\{z_{y_k}^{+}\exp[i\phi(\rho_{y_k}^{+})]+z_{y_k}^{-}\exp[-i\phi(\rho_{y_k}^{-})]\})]\,d\mu_\beta(\phi) \tag{4.6}$$

where

$$\{\mathcal{N}_{(k,y,r)},(\rho_y,z_y)\}_{y\in\Lambda^{(k)}-\{0,x_k\}}\cup\{\mathscr{E}_{(k,y_k,r,s)},(\rho_{y_k}^{\pm},z_{y_k}^{\pm})\}_{y_k=0,x_k}$$

is a (k, r, s)-modified charge assignment; w_k is the modified external charge density which is localized on $B_k(0)\cup B_k(x_k)$ for $k\leqslant N_0$ and satisfies

$$\sum_{y\in B_k(0)} w_k(y)=-\sum_{y\in B_k(x_k)} w_k(y)=1$$

and for $k>N_0$, w_k is localized on $B_k(0)$ with $Q(w_k)=0$. The function $W^{(k)}=W^{(k)}(\delta,\beta,\xi)$ is the falloff factor given by

(i) $W^{(1)}=1$.

(ii)

$$\left(\frac{L_k}{3L_{k-1}}\right)^{-\beta\delta\eta^2/4} W^{(k-1)}\leqslant W^{(k)}\leqslant\left(\frac{L_k}{3L_{k-1}}\right)^{-\beta\delta\eta^2/5} W^{(k-1)} \quad \text{for } k=2,\dots,N_0 \tag{4.7}$$

(iii) $W^{(k)}=[W^{(N_0)}]^2$ for $k\geqslant N_0+1$.

In this language, Theorem 4.2 states that, for a choice of parameters such that we have

$$K_1(L_1,\beta)\leqslant\frac{1}{L_1^{r+2(\alpha-1)}} \quad\text{and}\quad K_1'(L_1,\beta,\xi)<\frac{1}{L_1^s} \tag{4.8}$$

simultaneously, the two-point external charges partition function $Z_{\xi,\Lambda}$ is a convex combination of $(1, r, s)$-regular external charges partition functions.

Thus, Theorem 4.2 gives us the initial step in the inductive procedure for the following theorem.

Theorem 4.3. Let $3/2<\alpha<2$, $2\alpha(\alpha-1)/(2-\alpha)<r<\beta\delta/2-2\alpha$, $0<s<\min\{(1/\alpha)[r-\frac{1}{4}\beta\delta(\alpha-1)\eta^2], \frac{1}{4}\beta\delta\bar{\eta}^2\}$ with $\delta^{-1}=8+O(L_1^{3-2\alpha})$. Suppose (4.8) holds; then, if L_1 is large enough, the two-point external charges partition function $Z_{\xi,\Lambda}$ can always be written as a convex-combination of (k, r, s)-regular partition functions for $k=1, 2,..., N$.

In view of Theorem 4.2, Theorem 4.3 is consequence of the following lemma.

Lemma 4.4. Let α, r, s, L_1 be as above. Let $k=1, 2,..., N-1$. Then if L_1 is large enough, any (k, r, s)-regular two-point external charges partition function can be written as a convex combination of $(k+1, r, s)$-regular external charges partition functions.

Proof of Lemma 4.4. (Part A). Let $k=1,..., N_0-1$. Let $Z_{\xi,\Lambda}^{(k,r,s)}$ be given by (4.5). As in Section 3, for all $y\in\Lambda^{(k+1)}-\{0, x_{k+1}\}$ we define

$$\mathcal{N}_{(k+1, y, r)}=\bigcup_{u\in B_{k+1}^{(k)}(y)}\mathcal{N}_{(k, u, r)}$$

and for $y_{k+1}=0, x_{k+1}$

$$\tilde{\mathscr{E}}_{(k+1, y_{k+1}, r, s)}=\bigcup_{\substack{u\in B_{k+1}^{(k)}(y_{k+1})\\ u\neq y_k}}\mathcal{N}_{(k, u, r)}\cup\mathscr{E}_{(k, y_k, r, s)}$$

Clearly they are $(k+1, y, r)$ and $(k+1, y_{k+1}, r, s)$-sparse neutral ensembles, respectively, and

$$\begin{aligned}Z_{\xi,\Lambda}^{(k,r,s)}(x)=\int\{\exp[i\xi\phi(w_k)]\}&\prod_{\substack{u\in\Lambda^{(k+1)}\\ u\neq 0, x_{k+1}}}F(\tilde{\eta}_{(k+1, u, r)};\phi)\\ &\times\prod_{y\in B_{k+1}^{(k)}(u)}[1+z_y\cos\phi(\rho_y)]\\ &\times\prod_{y_{k+1}=0, x_{k+1}}\Bigg[W^{(k)}G(\tilde{\mathscr{E}}_{(k+1, y_{k+1}, r, s)};\phi)\\ &\times\prod_{\substack{y\in B_{k+1}^{(k)}(y_{k+1})\\ y\neq y_k}}[1+z_y\cos\phi(\rho_y)]\\ &\times\left(1+\tfrac{1}{2}\{z_{y_k}^{+}\exp[i\phi(\rho_{y_k}^{+})]+z_{y_k}^{-}\exp[-i\phi(\rho_{y_k}^{-})]\}\right)\Bigg]d\mu_\beta(\phi)\end{aligned}\tag{4.9}$$

Using Lemma 2.1 for $u\in\Lambda^{(k+1)}-\{0, x_{k+1}\}$ and Lemma 2.1 combined with

Lemma 4.1 for $y_{k+1}=0,\ x_{k+1}$, we can write this as a convex continuation of partition functions of the form

$$\int \{\exp[i\xi\phi(w_k)]\} \prod_{\substack{u\in\Lambda^{(k+1)}\\ u\neq 0, x_{k+1}}} \{F(\tilde{\mathcal{N}}_{(k+1,u,r,s)};\phi)[1+z_u^*\cos\phi(\rho_u^*)]\}$$
$$\times \prod_{y_{k+1}=0,x_{k+1}} [W^{(k)}G(\tilde{\mathscr{E}}_{(k+1,y_{k+1},r,s)};\phi)$$
$$\times(1+\tfrac{1}{2}\{z_{y_{k+1}}^+\exp[i\phi(\rho_{y_{k+1}}^+)]+z_{y_{k+1}}^-\exp[-i\phi(\rho_{y_{k+1}}^-)]\})]\,d\mu_\beta(\phi)$$

where (ρ_u^*, z_u^*) for $u\in\Lambda^{(k+1)}-\{0, x_{k+1}\}$ is given by (3.3) and

$$\rho_{y_{k+1}}^{\pm}=\tau_1\rho_\sigma+\tau_2\rho_{y_k}^{\pm}$$

with

$$\rho_\sigma=\sum_{\substack{y\in B_{k+1}^{(k)}(y_{k+1})\\ y\neq y_k}}\sigma_y\rho_y$$

for some $\sigma\in\mathscr{G}(B_{k+1}^{(k)}(y_{k+1})-\{y_k\})$ and

$$z_{y_{k+1}}^{\pm}\leqslant\left[\frac{2}{\log 2}z_\sigma\right]^{|\tau_1|}\left[\frac{2}{\log 2}z_{y_k}^{\pm}\right]^{|\tau_2|}$$

with

$$z_\sigma\leqslant\left[\frac{1}{\log 2}\frac{1}{L_k^r}\right]^{\Sigma_y|\sigma_y|}$$

To propagate our bound on the activity, we need to apply Lemma 3.3, in some cases, to extract a self-energy term. As in Section 3, Lemma 3.3 requires (3.4), which can be established applying Lemma 2.1 to those ρ_u^* such that $\rho_u^*=\rho_y$ for some $y\in B_{k+1}^{(k)}(u)$, $u\neq 0,\ x_{k+1}$. In the case $\rho_{y_{k+1}}^{\pm}=\rho_{y_k}^{\pm}$, we apply Lemma 4.1 to obtain (3.4), but now we notice that the resulting density $\rho_{y_{k+1}}^{\pm}$ has support on $B(y_{k+1}, (10/3)\,L_{k+1})\equiv\hat{B}_{k+1}(y_{k+1})$. Therefore, to say $\rho_y^{\pm}$ is locatized on $B_k(y_k)$ will mean $\operatorname{supp}\rho_y^{\pm}\subset\hat{B}_k(y_k)$.

It follows that $Z_{\xi,\Lambda}^{(k,r,s)}$ can be written as a convex combination of terms of the form

$$\int \{\exp[i\xi\phi(w_k)]\} \prod_{\substack{u\in\Lambda^{(k+1)}\\ u\neq 0, x_{k+1}}} F(\mathcal{N}_{(k+1,u,r)};\phi)[1+z_u\cos\phi(\rho_u)]$$
$$\times \prod_{y_{k+1}=0,x_{k+1}} [W^{(k)}G(\tilde{\mathscr{E}}_{(k+1,y_{k+1},r,s)};\phi)$$
$$\times(1+\tfrac{1}{2}\{z_{y_{k+1}}^{+*}\exp[i\phi(\rho_{y_{k+1}}^{+*})]+z_{y_{k+1}}^{-*}\exp[-i\phi(\rho_{y_{k+1}}^{-*})]\})]\,d\mu_\beta(\phi) \tag{4.10}$$

where

$$\{\mathcal{N}_{(k+1,u,r)}, (\rho_u, z_u)\}_{u\in\Lambda^{(k+1)}-\{0,x_{k+1}\}}$$

is a $(k+1, r)$-regular charge assignment and $(\rho^{\pm *}_{y_{k+1}}, z^{\pm *}_{y_{k+1}})$ is a weighted charge localized on $B_{k+1}(y_{k+1})$ with

$$z^{\pm *}_{y_{k+1}} \leqslant \left[\frac{2}{\log 2} z_\sigma\right]^{|\tau_1|} \left[\left(\frac{2}{\log 2}\right)^4 z^{\pm}_{y_k}\right]^{|\tau_2|} \tag{4.10a}$$

and

$$z_\sigma \leqslant \left[\frac{2^3}{(\log 2)^4} \frac{1}{L^r_k}\right]^{\Sigma_y |\sigma_y|} \tag{4.10b}$$

for some $\sigma \in \overline{\mathscr{G}(B^{(k)}_{k+1}(y_{k+1})} - \{y_{k+1}\})$.

We now consider several cases.

(i) $|\tau_1| \neq 0$. Define

$$\mathscr{E}_{(k+1, y_{k+1}, r, s)} = \tilde{\mathscr{E}}_{(k+1, y_{k+1}, r, s)}$$

Now we distinguish each factor of

$$\{\exp[i\xi\phi(w_k)]\}\ W^{(k)}(1 + \tfrac{1}{2}\{z^+_{y_{k+1}} \exp[i\phi(\rho^{+*}_{y_{k+1}})] + z^-_{y_{k+1}} \exp[i\phi(\rho^-_{y_{k+1}})]\}$$

In the former we use Lemma 3.3 to replace w_k by $w_{k+1} = \overline{w_k}$ and $W^{(k)}$ by $W^{(k+1)}$ (here we chose $\kappa = \eta^2\delta/4\xi^2$). In the latter factor we notice that (4.7) and (4.10) give

$$\frac{W^{(k)}}{W^{(k+1)}} z^{\pm *}_{y_{k+1}} \leqslant \left(\frac{L_{k+1}}{3L_k}\right)^{\beta\delta\eta^2/4} \frac{2^4}{(\log 2)^5} L^{-r}_k \leqslant L^{-s}_{k+1}$$

if $s < (1/\alpha)[r - \frac{1}{4}\beta\delta(\alpha-1)\eta^2]$ and L_1 sufficiently large.

Thus, in case (i) we can replace

$$\{\exp[i\xi\phi(w_k)]\} \prod_{y_{k+1}=0,x_{k+1}} [W^{(k)}(1 + \tfrac{1}{2}\{z^{+*}_{y_{k+1}} \exp[i\phi(\rho^{+*}_{y_{k+1}})] + z^{-*}_{y_{k+1}} \exp[-i\phi(\rho^{-*}_{y_{k+1}})]\})] \tag{4.11}$$

by

$$\exp[i\xi\phi(w_{k+1})] \prod_{y_{k+1}=0,x_{k+1}} [W^{(k+1)}(1 + \tfrac{1}{2}\{z^+_{y_{k+1}} \exp[i\phi(\rho^+_{y_{k+1}})] + z^-_{y_{k+1}} \exp[-i\phi(\rho^-_{y_{k+1}})]\})] \tag{4.12}$$

where

$$(\rho^{\pm}_{y_{k+1}}, z^{\pm}_{y_{k+1}}) = \left(\tilde{\rho}^{\pm}_{y_{k+1}}, z^{\pm *}_{y_{k+1}} \frac{W^{(k)}}{W^{(k+1)}}\right)$$

is a weighted charge density such that

$$\tilde{\rho}^{\pm}_{y_{k+1}} = \rho^{\pm *}_{y_{k+1}} \pm (w_k - w_{k+1}) \mid_{B_{k+1}(y_{k+1})}$$

is localized on $B_{k+1}(y_{k+1})$. Notice that $Q(\tilde{\rho}^{\pm}_{y_{k+1}}) = Q(\rho^{\pm *}_{y_{k+1}})$.

(ii) $|\tau_1| = 0$. In this case $\rho^{\pm *}_{y_{k+1}} = \rho^{\pm}_{y_k}$. Here we must consider two subcases:

(iia) $Q(\rho^{\pm}_{y_k}) = 0$. We define

$$\mathscr{E}_{(k+1, y_{k+1}, r, s)} = \tilde{\mathscr{E}}_{(k+1, y_{k+1}, r, s)} \cup \{(\rho^{\pm *}_{y_{k+1}}, z^{\pm *}_{y_{k+1}})\}$$

and notice that it is a $(k+1, y_{k+1}, r, s)$-sparse neutral ensemble by (4.10a) and (410.b).

(iib) $Q(\rho^{\pm}_{y_k}) \neq 0$. We define

$$\mathscr{E}_{(k+1, y_{k+1}, r, s)} = \tilde{\mathscr{E}}_{(k+1, y_{k+1}, r, s)}$$

and we use Lemma 3.3 to replace (4.11) by (4.12) where now

$$(\rho^{\pm}_{y_{k+1}}, z^{\pm}_{y_{k+1}}) = \left(\tilde{\rho}^{\pm}_{y_{k+1}}, z^{\pm *}_{y_{k+1}} \frac{W^{(k)}}{W^{(k+1)}}\right)$$

with

$$\tilde{\rho}^{\pm}_{y_{k+1}} = \overline{\rho^{\pm *}_{y_{k1}} \pm w_k} \mid_{B_{k+1}(y_{k+1})} \pm \overline{w_k} \mid_{B_{k+1}(y_{k+1})}$$

which is a charge density localized on $B_{k+1}(y_{k+1})$ with

$$z^{\pm}_{y_{k+1}} \leqslant \left(\frac{L_{k+1}}{3L_k}\right)^{(\beta\delta/4)\eta^2 - (\beta\delta/2)(q \pm \xi)^2} z^{\pm *}_{y_{k+1}} \leqslant L^{-s}_{k+1}$$

if $s < \frac{1}{4}\beta\delta\bar{\eta}^2$ and L_1 sufficiently large.

This completes the proof of Lemma for $k \leqslant N_0 - 1$.

Part B. Now, let us assume the assumptions of Lemma 4.4 for $k = N_0$. By our choice of x, the two modified external charges w_{N_0} are localized on $B_{N_0+1}(0)$ so, for $y \in \Lambda^{(N_0+1)} - \{0\}$ we define

$$\tilde{\mathscr{N}}_{(N_0+1, y, r)} = \bigcup_{u \in B^{(N_0)}_{N_0+1}} \mathscr{N}_{(N_0, u, r)}$$

and

$$\tilde{\mathscr{E}}_{(N_0+1,0,r,s)} = \bigcup_{\substack{u \in B^{(N_0)}_{N_0+1}(0) \\ u \neq 0, x_{N_0}}} \mathscr{N}_{(N_0,u,r)} \bigcup_{y_{N_0} = 0, x_{N_0}} \mathscr{E}_{(N_0, y_{N_0}, r, s)}$$

Clearly they are, respectively, (N_0+1, y, r)- and $(N_0+1, 0, r, s)$-sparse neutral ensembles. $Z^{(N_0,r,s)}_{\xi,\Lambda}(x)$ can be expanded as previously, except now $y_{k+1}=0$ is the only different point in $\Lambda^{(N_0+1)}$. Thus, it can be written as a convex combination of terms of the form

$$\int \{\exp[i\xi\phi(w_{N_0})]\} \prod_{\substack{y \in \Lambda^{(N_0+1)} \\ y \neq 0}} F(\mathscr{N}_{(N_0+1,y,r)}; \phi)[1 + z_y \cos\phi(\rho_y)]$$
$$\times W^{(N_0)} G(\tilde{\mathscr{E}}_{(N_0+1,0,r,s)}; \phi)(1 + \tfrac{1}{2}\{z_0^{+*} \exp[i\phi(\rho_0^{+*})]$$
$$+ z_0^{-*} \exp[-i\phi(\rho_0^{-*})]\}) \, d\mu_\beta(\phi)$$

where

$$\{\mathscr{N}_{(N_0+1,y,r)}(\rho_y, z_y)\}_{y \in \Lambda^{(N_0+1)} - \{0\}}$$

is an (N_0+1, r)-regular charge assignment and $(\rho_0^{\pm *}, z_0^{\pm *})$ is a weighted charge density, obtained by a simple extension of Lemma 4.1, and given by

$$\rho_0^{\pm *} = \tau_1 \rho_\sigma + \tau_2 \rho_0^{\pm} + \tau_3 \rho^{\pm}_{x_{N_0}}$$

with $\tau_i = 0, \pm 1$, $i = 1, 2, 3$, $\tau \neq (0, 0, 0)$, and

$$\rho_\sigma = \sum_{\substack{y \in B^{N_0}_{N_0+1}(0) \\ y \neq 0, x_{N_0}}}$$

for some $\sigma \in \mathscr{G}(\overline{B^{(N_0)}_{N_0+1}}(0) - \{0, x_{N_0}\})$ and

$$z_0^{\pm *} \leqslant \left[\frac{3}{\log 2} z_\sigma\right]^{|\tau_1|} \left[\frac{3}{\log 2} z_0^{\pm}\right]^{|\tau_2|} \left[\frac{3.2^3}{(\log 2)^4} z^{\pm}_{x_{N_0}}\right]^{|\tau_3|}$$

with z_σ satisfying (4.10b).

Notice that we cannot extract any more decay from the external charges after the scale N_0. This makes the estimates on the activities easier than those considered in Case A, as we show below. We now have the following cases:

(i) $\tau_1 \neq 0$. As in Part A, we define $\mathscr{E}_{(N_0+1,0,r,s)} = \tilde{\mathscr{E}}_{(N_0+1,0,r,s)}$ and notice that

$$z_0^{\pm} \leqslant \frac{3 \cdot 2^3}{(\log 2)^5} \frac{1}{L_k^r} \leqslant L_{k+1}^{-s}$$

if $s < \frac{1}{2} r$ and L_1 sufficiently large.

(ii) $\tau_1 = 0$. In this case we have to consider three different cases:

(iia) τ_2 and $\tau_3 \neq 0$. Define $\mathscr{E}_{(N_0+1,0,r,s)}$ as in case (i) and notice that

$$z_0^{\pm *} \leqslant \frac{3^2 \cdot 2^3}{(\log 2)^5} \frac{1}{L_k^{2s}} \leqslant L_{k+1}^{-s}$$

if L_1 is large enough.

(iib) τ_2 or $\tau_3 \neq 0$ and $Q(\rho_0^{\pm *}) = 0$. Define $\mathscr{E}_{(N_0+1,0,r,s)} = \tilde{\mathscr{E}}_{(N_0+1,0,r,s)} \cup \{(\rho_0^{\pm *}, z_0^{\pm *})\}$ and notice that $\mathscr{E}_{(N_0+1,0,r,s)}$ is an $(N_0+1, 0, r, s)$-sparse modified neutral ensemble.

(iic) τ_2 or $\tau_3 \neq 0$ and $Q(\rho_{y_{N_0}}^{\pm *}) \neq 0$.

We define $\mathscr{E}_{(N_0+1,0,r,s)}$ as in case (i).

In this case we must use Lemma 3.3 judiciously to propagate the estimate on the activity. If we have

$$B(0, \tfrac{1}{3} L_{k+1}) \cap B_k(x_k) = \phi \tag{4.13}$$

we apply Lemma 3.3, which gives

$$z_{y_{k+1}}^{\pm *} \leqslant \frac{3 \cdot 2^2}{(\log 2)^4} \left(\frac{L_{k+1}}{3L_k}\right)^{-(\beta\delta/2)\bar{\eta}^2} L_k^{-s} \leqslant L_{k+1}^{-s}$$

if $s < (\beta\delta/2)\bar{\eta}^2$ and L_1 sufficiently large.

If condition (4.13) is not satisfied, let $B(y_k, CL_k)$ be the maximal square, where $\rho_{y_k}^{\pm}$ is localized, satisfying condition (3.4). By Lemma 3.3,

$$\overline{z_{y_{k+1}}^{\pm *}} \leqslant \frac{3 \cdot 2^3}{(\log 2)^4} L_k^{-s} \left(\frac{C}{3}\right)^{-(\beta\delta/2)\bar{\eta}^2}$$

This is sufficient to propagate the bound, unless (4.13) does not hold. In this case we skip the box $B_k(y_k)$ where the other external charge is localized and continue applying Lemma 3.3. This gives us

$$z_{y_{k+1}}^{\pm} \leqslant \frac{3 \cdot 2^2}{(\log 2)^4} L_k^{-s} \left(\frac{C}{3}\right)^{-(\beta\delta/2)\bar{\eta}^2} \left(\frac{L_{k+1}}{3(C+1)L_k}\right)^{-\beta\delta/2} \leqslant L_{k+1}^{-s}$$

if $s \leqslant (\beta\delta/4)\bar{\eta}^2$ and L_1 sufficiently large.

This completes the proof of Part B.

Part C. For $k = N_0 + 1, ..., N$, we simply repeat the steps of the proof of Part A, but the estimates go more easily.

5. THE EXTERNAL CHARGES CORRELATION FUNCTION

We are now ready to prove Theorems 1.1 and 1.2. Given $x \in \mathbf{Z}^2$ with $L_{N_0} < 2|x| < L_{N_0+1}$, let $\Lambda = [-R, R]^2$ with $L_{N-1} < R \leqslant L_N$, $N > N_0$. We pick $3/2 < \alpha < 2$, L_1 sufficiently large, and define $\beta_{\text{P.F.}} = \inf\{\beta \in \mathbf{R}^+ : J_\alpha(\beta) \neq \phi\}$, where

$$J_\alpha(\beta) = \left(\frac{2\alpha(\alpha-1)}{2-\alpha}, \frac{\beta\delta}{2} - 2\alpha\right) \tag{5.1}$$

For $\beta > \beta_{\text{P.F.}}$ we now pick r, $s > 0$ satisfying the assumptions of Theorem 4.3 and define β_I as the temperature which, for a choice of parameters in the *a priori* distribution λ, satisfies the initial condition for Theorem 4.3 [see (4.8)].

Thus, if we define the critical temperature $\bar{\beta} = \bar{\beta}(\lambda)$ by

$$\bar{\beta} = \max\{\beta_{\text{P.F.}}, \beta_I\}$$

Theorem 4.3 says that, for $\beta > \bar{\beta}$, the external charges correlation function $G_{\Lambda,\xi}(x)$ can be written as

$$\begin{aligned} G_{\Lambda,\xi}(x) &= \frac{\sum_{\gamma \in \mathscr{F}} c_\gamma Z^{(N,r,s)}_{\Lambda,\xi,\gamma}(x)}{\sum_{\gamma \in \mathscr{F}} c_\gamma Z^{(N,r,s)}_{\Lambda,0,\gamma}} \\ &\equiv \sum_{\gamma \in \mathscr{F}} d_\gamma \frac{Z^{(N,r,s)}_{\Lambda,\xi,\gamma}(x)}{Z^{(N,r,s)}_{\Lambda,0,\gamma}} \end{aligned} \tag{5.2}$$

where c_γ, $d_\gamma > 0$ with $\sum_\gamma c_\gamma = \sum_\gamma d_\gamma = 1$. For each $\gamma \in \mathscr{F}$, $Z^{(N,r,s)}_{\Lambda,\xi,\gamma}(x)$ is an (N, r, s)-regular external charges partition function and $Z^{(N,r,s)}_{\Lambda,0,\gamma}$ is the same expression with $\xi = 0$.

Our choice for the denominators in (5.2) is important to the cancellations we will perform between the numerators and denominators.

Let us first point out the properties of $Z^{(N,r,s)}_{\Lambda,0,\gamma}$ which differ from those of $Z^{(N,r,s)}_{\Lambda,\xi,\gamma}$ with $\xi \neq 0$. Clearly:

(i) The phase term $e^{i\xi\phi(w_k)}$ in (4.5) is dropped.

(ii) $W^{(k)} = 1$ for $k = 1, 2, ..., N$.

(iii) And, the most important, for $k = 1, ..., N$,

$$\rho^+_{y_k} = \rho^-_{y_k} \equiv \rho_{y_k} \qquad \text{and} \qquad z^+_{y_k} = z^-_{y_k} \equiv z_{y_k}$$

Therefore

$$1+\tfrac{1}{2}\{z_{y_k}^{+}\exp[i\phi(\rho_{y_k}^{+})]+z_{y_k}^{-}\exp[-i\phi(\rho_{y_k}^{-})]\}=1+z_{y_k}\cos\phi(\rho_{y_k})$$

where (ρ_{y_k}, z_{y_k}) is a (k, s)-admissible weighted charge density, and the functional $G(\mathscr{E}_{(N,0,r,s)}; \phi)$ is real and strictly positive.

Since the characteristic function of $d\mu_\beta(\phi)$ is a strictly positive function, it follows from (iii) that $Z_{\Lambda,0,\gamma}^{(N,r,s)} \geqslant 1 > 0$ and (5.2) is well defined.

We now notice that the number of modified weighted charge densities $(\rho^{\pm}, z^{\pm})$ in $\mathscr{E}_{(N,0,r,s)}$ is at most $2N$ (actually it is at most $N+N_0$). We also notice the simple bound

$$1-cL_k^{-s}\leqslant|1+\tfrac{1}{2}\{x_{y_k}^{+}\exp[i\phi(\rho_{y_k}^{+})]+z_{y_k}^{-}\exp[-i\phi(\rho_{y_k}^{-})]\}|\leqslant 1+cL_k^{-s} \tag{5.3}$$

with $c=2^3\cdot 3/(\log 2)^4$, which is valid without the modulus when $\xi=0$.

From (4.5), properties (i)–(iii) above, and (5.3) we have

$$|Z_{\Lambda,\xi,\gamma}^{(N,r,s)}(x)|\leqslant\prod_{k=1}^{N}(1+cL_k^{-s})^2\,W^{(N)}\int d\mu_\beta(\phi)\,G(\tilde{\mathscr{E}}^{\gamma}_{(N,0,r,s)};\phi)$$

and

$$Z_{\Lambda,0,\gamma}^{(N,r,s)}(x)\geqslant\prod_{k=1}^{N}(1-cL_k^{-s})^2\int d\mu_\beta(\phi)\,G(\tilde{\mathscr{E}}^{\gamma}_{(N,0,r,s)};\phi)$$

where $\tilde{\mathscr{E}}^{\gamma}_{(N,0,r,s)}$ is the $(N, 0, r, s)$-sparse neutral ensemble defined by induction:

(i) For $k=1$, $\tilde{\mathscr{E}}_{(1,y_1,r,s)}=\phi$.

(ii) For $k=2, 3,.., N_0$ and $k=N_0+2,..., N$

$$\tilde{\mathscr{E}}_{(k,y_k,r,s)}=\bigcup_{\substack{u\in B_k^{(k+1)}(y_k)\\ u\neq y_{k-1}}}\mathscr{N}_{(k-1,u,r)}\cup\tilde{\mathscr{E}}_{(k-1,y_{k-1},r,s)}$$

and (iii)

$$\tilde{\mathscr{E}}_{(N_0+1,0,r,s)}=\bigcup_{\substack{u\in B_{N_0+1}^{(N_0)}(0)\\ y\neq 0,x_{N_0}}}\mathscr{N}_{(N_0,u,r)}\bigcup_{y_{N_0}=0,x_{N_0}}\tilde{\mathscr{E}}_{(N_0,y_{N_0},r,s)}$$

where each $\mathscr{N}_{(k-1,u,r)}$ is a $(k-1, u, r)$-sparse neutral ensemble as introduced in Section 3.

Thus,

$$\left|\frac{Z_{\Lambda,\xi,\gamma}^{(N,r,s)}(x)}{Z_{\Lambda,0,\gamma}^{(N,r,s)}}\right| \leqslant C \prod_{k=2}^{N_0} \left(\frac{L_k}{3L_{k-1}}\right)^{-2\beta\delta\eta^2/5} \leqslant C\left(\frac{L_{N_0}}{3^{N_0-1}L_1}\right)^{-2\beta\delta\eta^2/5} \leqslant C\,|x|^{-\beta\theta\eta^2}$$

This completes the proof of Theorem 1.1.

Now let us restrict our attention to the hard-core and standard gases. In these models the initial condition for Theorem 4.3 can be satisfied by simply taking the activity z sufficiently small and this implies $\bar{\beta} = \beta_{\text{P.F.}}$. From (5.1) the critical temperature is given by

$$\bar{\beta} = \frac{4\alpha}{\delta(2-\alpha)} > \frac{12}{\delta}$$

Theorem 1.2 follows if we choose the best estimate for $\delta^{-1} = 2\pi + O(L_1^{3-2\alpha})$ and notice that α can be chosen arbitrarily close to 3/2 and L_1 can be chosen arbitrarily large, taking z arbitrarily small.

APPENDIX A. PROOF OF LEMMA 2.1

We have

$$\prod_{i\in I} (1 + z_i \cos\phi_i) = 1 + \sum_{n=1}^{N} \sum_{\substack{\{i_k \in I\}_{k=1}^n \\ i_1 \neq i_2 \neq \cdots \neq i_n}} z_{i_1} \cdots z_{i_n} \cos\phi_{i_1} \cdots \cos\phi_{i_n} \tag{A.1}$$

Using

$$\cos\phi_{i_1} \cos\phi_{i_2} \cdots \cos\phi_{i_n} = \sum_{\{\sigma_{i_k} = \pm 1\}_{k=1}^n} \frac{1}{2^n} \cos(\sigma_{i_1}\phi_{i_1} + \cdots + \sigma_{i_n}\phi_{i_n})$$

(A.1) can be written as

$$1 + \sum_{\sigma\in\mathscr{G}(I)} \prod_{i\in I} \left(\frac{z_i}{2}\right)^{|\sigma_i|} \cos\phi(\sigma) = 1 + \sum_{\sigma\in\mathscr{G}(I)} \prod_{i\in I} \left(\frac{1}{2b_I}\right)^{|\sigma_i|} z_\sigma \cos\phi(\sigma) \tag{A.2}$$

where $\mathscr{G}(I) = \{\sigma: I \to \{0, +1, -1\}, \sigma \text{ not identically zero}\}$,

$$\phi(\sigma) = \sum_{i\in I} \sigma_i \phi_i$$

$$z_\sigma = \prod_{i\in I} (b_I z_i)^{|\sigma_i|}$$

In the expression (A.2) we multiplied and divided each term in the sum by $\prod_{i\in I} b_I^{|\sigma_i|}$, with b_I to be fixed below.

Hence,

$$\prod_{i \in I} (1 + z_i \cos \phi_i) = \sum_{\sigma \in \mathscr{G}(I)} c_\sigma [1 + z_\sigma \cos \phi(\sigma)]$$

since $c_\sigma = \prod_i (1/2b_I)^{|\sigma_i|}$ satisfies

$$\sum_{\sigma \in \mathscr{G}(I)} c_\sigma = 1 \tag{A.3}$$

The above equations determine uniquely the constant b_I, i.e.,

$$1 = \sum_{n=1}^{N} \binom{N}{n} \frac{1}{b_I^n} = \left(1 + \frac{1}{b_I}\right)^N - 1$$

or

$$\ln 2 = N \ln \left(1 + \frac{1}{b_I}\right) \leqslant \frac{N}{b_I}$$

which proves the lemma.

APPENDIX B. PROOF OF LEMMA 3.3

Suppose for simplicity that $\rho_{u_0}^* = \rho_0^*$ is localized on $B_k(0)$. By hypothesis, $\operatorname{supp} \rho_u^* \cap B(0, \frac{1}{3} L_{k+1}) = \phi$ for all $u \in \Lambda^{(k+1)}$, $u \neq 0$, and we pick $n_k < D \leqslant n_{k+1} - 1$.

For each density $\rho_y \in \tilde{\mathscr{N}}_{(k+1,0,r)}$ let $B_{n_y}(y)$ be the box where ρ_y is localized, with

$$n_y = \inf\{n \in \mathbf{N}: \operatorname{supp} \rho_y \subset \overline{B_n(y)}\}$$

and let us introduce the scarce set $K_D(0) \subset B_{k+1}(0)$ given by

$$K_D(0) = \bigcup_{\substack{y \in B_{k+1}(0): \\ \rho_y \in \tilde{\mathscr{N}}_{(k+1,0,r)}}} \overline{B_{n_y}(y)}$$

Now we define the equivalence

$$y \sim y' \Leftrightarrow \overline{B_{n_y(y)}} \cap \overline{B_{n_{y'}}(y')} \neq \phi$$

and let $Y_1, \ldots, Y_P$ denote the distinct equivalence class.

Clearly, $K_D(0)$ can be written as the union of the $B(Y_t)$, $t = 1, \ldots, P$, where

$$B(Y_t) = \bigcup_{y \in Y_t} \overline{B_{n_y}}(y)$$

We have (see the definition of sparse ensembles $\mathcal{N}$)

$$B(Y) \subset B_{n_{\bar{y}}+1}(\bar{y}) \equiv \bar{B}(Y) \tag{B.1}$$

for some $\bar{y}$ such that $\bar{y} \in Y$ and $n_{\bar{y}} = \max\{n_y\}_{y \in Y}$.

Now we pick a real-valued function on $\mathbf{Z}^2$ given by

$$f_k(j) = \begin{cases} \log \dfrac{3^D}{L_k} & \text{for} \quad \|j\| \leqslant \dfrac{L_k}{2} \\ \log \dfrac{3^D}{2\,\|j\|} & \text{for} \quad \dfrac{L_k}{2} \leqslant \|j\| \leqslant \dfrac{3^D}{2} \\ 0 & \text{for} \quad \|j\| > \dfrac{3^D}{2} \end{cases} \tag{B.2}$$

and define the function

$$\hat{f}_k(j) = \begin{cases} f_k(j) & \text{if} \quad j \notin \bigcup_{t=1}^{P} \overline{B(Y_t)} \\ f_k(\bar{y}_t) & \text{if} \quad j \in \overline{B(Y_t)}, \quad t = 1,..., P \end{cases}$$

In order to prove identity (3.5), we perform the following complex shift in the ϕ variable:

$$\phi \to \phi' = \phi + i\beta\kappa q \hat{f}_k$$

where κ is a parameter to be chosen later and $q = Q(\rho_0^*)$.

Clearly [see (3.4)]

$$\phi'(\rho_u^*) = \phi(\rho_u^*) \qquad \text{for} \quad u \neq 0$$

and because of the neutrality of all $\rho \in \tilde{\mathcal{N}}_{(k+1,u,r)}$ and definition of $\hat{f}_k$,

$$F(\tilde{\mathcal{N}}_{(k+1,u,r)}; \phi') = F(\tilde{\mathcal{N}}_{(k+1,u,r)}; \phi), \forall u \in \Lambda^{(k+1)}$$

Thus, the only terms affected by the shift are

$$\begin{aligned} e^{i\phi'(\rho_0^*)} &= e^{i\phi(\rho_0^*)} e^{-\beta\kappa q f_k(\rho_0^*)} \\ &= e^{i\phi(\rho_0^*)} \left(\frac{3^D}{L_k}\right)^{-\beta\kappa q^2} \end{aligned}$$

and

$$d\mu_\beta(\phi') = d\mu_\beta(\phi)\, e^{i\kappa q \phi(\Delta \hat{f}_k)} e^{\beta\kappa^2 q^2 (\hat{f}_k, -\Delta \hat{f}_k)/2}$$

Define $\overline{\rho_0^*} = \rho_0^* + \kappa q(\Delta \hat{f}_k)$; Lemma 3.3 follows if

$$0 \leqslant (\hat{f}_k, -\Delta \hat{f}_k) \leqslant \delta \log \frac{3^D}{L_k}$$

for some constant $\delta < \infty$.

We notice that

$$(\hat{f}_k, -\Delta \hat{f}_k) \leqslant (f_k, -\Delta f_k) + E \tag{B.3}$$

with

$$E = \sum_{t=1}^{P} \sum_{j \in \partial \overline{B(Y_t)}} [f_k(j) - f_k(\bar{y}_t)]^2$$

where by $\partial \bar{B}$ we mean the boundary of the box $\bar{B}$.

The first term of the rhs of (B.3) is given by

$$= 4 \sum_{j = L_k/2}^{3^D/2} 2j \ln\left(\frac{j+1}{j}\right) \leqslant \ln\left(\frac{3^D}{L_k}\right)^8$$

At this point we could replace 8 by 2π if we had used the usual Euclidean norm instead the sup norm in (B.2).

We estimate the second term as follow: Let us decompose $B_{3^D}(0) \backslash B_k(0)$ in mutually disjoint, concentric regions:

$$\begin{aligned} B_{3^D}(0) \backslash B_k(0) &= \bigcup_{n = n_k + 1}^{D} B_{3^n}(0) \backslash B_{3^{n-1}}(0) \\ &\equiv \bigcup_{n = n_k + 1}^{D} A_n(0) \end{aligned}$$

and use the bound

$$|f_k(j+1) - f_k(j)| \leqslant \frac{2}{3^{n-1}} \qquad \text{for} \quad j \in A_n(0)$$

Thus (for details see ref. 10)

$$E \leqslant \sum_{n = n_k + 1}^{D} \sum_{t: \bar{y}_t \in A_n} 4 \cdot 3^{3(n_{\bar{y}_t} + 1)} \left(\frac{4}{3^{2(n-1)}}\right)^2$$

To perform the summation in t, we divide $\{B(Y_t)\}$ into subsets according to its scale. Now we notice that (see definition of $\mathcal{N}$):

$$N_n^{(m)} \equiv \operatorname{card}\{Y_t; B(Y_t) \subset A_n \text{ and } m < n_{\bar{y}_t} < m+1\} \leqslant 3^{2n - 2\alpha m}$$

Therefore

$$E \leqslant 16 \cdot 3^5 \sum_{n=n_k+1}^{D} \sum_{m=n_1}^{\infty} 3^{(3-2\alpha)m} = 16 \cdot 3^5(D - n_k - 1)\, CL_1^{3-2\alpha}$$

with $C = C(L_1, \alpha) \to 1$ as $L_1 \to \infty$.

We conclude Lemma 3.3 by taking $\delta^{-1} = 8 + 16 \cdot 3^5 CL_1^{3-2\alpha}$.

Note. After the completion of this work we received ref. 11, which is also concerned with the two-dimensional lattice Coulomb gases and the Kosterlitz–Thouless transition.

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REFERENCES

1. J. Fröhlich and T. Spencer, Phase Diagrams and Critical Properties of (Classical) Coulomb Systems, Erice Lectures (1980).
2. D. Brydges and P. Federbush, *Commun. Math. Phys.* **73**:197 (1980).
3. J. Fröhlich and T. Spencer, *Commun. Math. Phys.* **81**:525 (1981).
4. D. H. U. Marchetti and J. F. Perez, *Phys. Lett. A* **778**:74 (1986).
5. D. H. U. Marchetti and J. F. Perez, *J. Stat. Phys.* **55**:141 (1989).
6. H. von Dreifus, On the effects of randomness in ferromagnetic models and Schrödinger operators, Ph.D. thesis, New York University (1987).
7. T. Spencer, *J. Stat. Phys.* **51**:1009 (1988).
8. H. von Dreifus and A. Klein, *Commun. Math. Phys.* **124**:285 (1989).
9. Wei-Shih Yong, *J. Stat. Phys.* **49**:1 (1987).
10. J. Fröhlich and T. Spencer, The Berezinskii–Kosterlitz–Thouless transition (energy–entropy arguments and renormalization in defect gases), in *Progress in Physics*, Vol. 7 (1983).
11. G. A. Braga: The two dimensional lattice Coulomb gas at low temperature, Ph.D. thesis, New York University (1989).